TURBULENT CONVECTION IN A VERTICAL TUBE

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Integral formulas expressing the theorems of momentum and kinetic energy for the case of combined forced and free convection in a vertical tube are obtained. These formulas can be used to calculate the velocity distribution in the cross section of the tube in the case of laminar, turbulent, and transitional flow regimes in the presence and absence of internal heat sources in the liquid arbitrarily distributed over the cross section. Integral formulas are derived for the determination of the drag coefficient and heat transfer; these are also valid for all flow regimes. The general formulas are used for the calculation of specific cases. Turbulent viscosity in the case of combined forced and free convection is discussed.

NOTATION

$v^{(0)}$ is the velocity of forced convection; $v^{(1)}$ is the velocity of free convection; $\langle v \rangle$ is the mean velocity over cross section of tube; $T^{(0)}$ is the temperature for forced convection; $T^{(1)}$ is the temperature for free convection; T_W is the temperature of tube wall; r_0 is the tube radius; ν_t is the turbulent viscosity; χ_t is the turbulent thermal diffusivity; A is the constant axial temperature gradient on tube wall; p_0 is the averaged pressure, corresponding to constant liquid temperature; y is the distance from tube wall; y_* is the dimensionless distance from wall; r is the distance from axis; Q is the quantity of heat produced by internal sources in unit volume of liquid in unit time; R^* is the Rayleigh number; z is the coordinate along tube axis, directed upward;

$$u^{(0)} \equiv \frac{v^{(0)}}{v_*}, \quad u^{(1)} \equiv \frac{v^{(1)}}{v_*}, \quad v^{(0,1)} = v^{(0)} + v^{(1)}, \quad u^{(0,1)} \equiv u^{(0)} + u^{(1)},$$

$$w^{(0)} \equiv \frac{v^{(0)}}{\langle v \rangle}, \quad w^{(1)} \equiv \frac{v^{(1)}}{\langle v \rangle}, \quad w^{(0,1)} \equiv w^{(0)} + w^{(1)}, \quad T^{(0,1)} = T^{(0)} + T^{(1)},$$

$$\theta \equiv \frac{T}{Ar_0}, \quad \xi \equiv \frac{r}{r_0}, \quad \psi \equiv 1 + \frac{\nu_t}{\nu}, \quad \psi_1 \equiv 1 + \frac{\chi_t}{\chi},$$

$$G \equiv \frac{g\beta A r_0^4}{\nu^2}, \quad P \equiv \frac{\nu}{\chi}, \quad P_t \equiv \frac{\nu_t}{\chi_t}, \quad P_1 \equiv \frac{P}{P_t},$$

$$R^* \equiv GP, \quad R_* \equiv \frac{v_* r_0}{\nu}, \quad R \equiv \frac{\langle v \rangle 2r_0}{\nu}, \quad Q_* \equiv \frac{Q}{\rho_0 C_p A v_*},$$

$$y_* = \frac{v_* y}{\nu} = R_* (1 - \xi),$$

$$\langle T \rangle - T_w \equiv \left[\int_0^1 (T - T_w) v \xi d\xi\right] \left(\int_0^1 v \xi d\xi\right)^{-1},$$

$$J_1(\xi) \equiv \int_1^\xi u^{(0)} \xi d\xi, \quad J_2(\xi) \equiv \int_0^\xi u^{(0,1)} \xi d\xi,$$

$$J_3(\xi) \equiv \int_0^\xi (u^{(0,1)} - Q_*) \xi d\xi,$$

$$J_4(\xi, 1) \equiv \int_1^\xi \frac{J_3(\xi)\, d\xi}{(1 + \chi_t / \chi)\, \xi}, \quad J_5(\xi, 0) \equiv \int_0^\xi \frac{J_3(\xi)}{1 + \chi_t / \chi} \xi d\xi,$$

$$J_6(\xi) \equiv \int_0^\xi w^{(0,1)} \xi d\xi,$$

$$J_7(\xi) \equiv \int_0^\xi (w^{(0,1)} - 1) \xi d\xi, \quad J_8(\xi) \equiv \int_0^\xi u^{(1)} \xi d\xi.$$

The mean temperature head is $\langle T \rangle - T_W$. The dynamic velocity v_* is determined from the equation $2\rho_0 v_*^2 / r_0 = -\rho_0 g - \partial p_0 / \partial z$.

1. Equations of problem. Turbulent viscosity. We consider a steady-state turbulent movement of liquid in a vertical round tube in which the wall temperature varies linearly along the tube. Let there also be a constant vertical pressure gradient and internal heat sources distributed arbitrarily, but axisymmetrically, over the cross section. The liquid is assumed to be mechanically incompressible, but thermally strained, so that free convection is superimposed on the forced flow. The flow is assumed to be axisymmetrical and the averaged velocity vertical.

With these assumptions the equations of the problem will be [1, 2]

$$\nabla (\psi \nabla u) = -2R_* - (G / R_*)\, \theta,$$
$$\nabla (\psi_1 \nabla \theta) = -PR_* Q_* + PR_* u. \qquad (1.1)$$

Equations (1.1) contain four unknown functions: u, θ, ν_t, χ_t. The two equations (1.1) are not sufficient for their determination. Hence, we find ν_t and χ_t by analysis of experimental data and from some physical considerations. The boundary condition is

$$u = 0 \text{ when } \xi = 1. \qquad (1.2)$$

The temperature in the case of mixed convection must be measured relative to a specially chosen mean calorimetric temperature of the liquid.

If the functions ν_t and χ_t are known, then Eqs. (1.1) can be used to find all the dynamic and thermal characteristics of turbulent convection in a vertical tube.

In the case of pure forced convection the turbulent viscosity close to the tube wall is expressed satisfactorily by the function [3]

$$\nu_t / \nu = 4.4 \quad ({}^1/_{11} y_* - \operatorname{th} {}^1/_{11}\, y_*) \qquad (1.3)$$

and in the core of the flow by the function

$$\nu_t / \nu = {}^1/_{15} R_* (1 - \xi^2)(1 + 2\xi^2) - 1. \qquad (1.4)$$

Near the wall (1.4) becomes meaningless. Formulas (1.3) and (1.4) differ from the corresponding formulas obtained in [3] in that for brevity the value of $\varkappa$ is substituted ($\varkappa = 0.4$), and in (1.3) the coefficient 11 is taken out of the brackets.

In the layer near the wall we take ν_t/ν according to (1.3) up to the point of conjunction with (1.4), and in the core of the flow according to (1.4). In Fig. 1 the curves 1, 2, and 3 represent (1.3), (1.4), and $\nu_t/\nu + 1$, respectively, for $R_* = 314.25$. A Reynolds number $R = 10^4$ corresponds to the value of the parameter $R_* = 314.25$ in the case of pure forced convection. We will regard the ratio $\nu_t/\chi_t \equiv P_t$ as constant.

The situation is more complicated in the case of combined forced and free turbulent convection. The shape of the velocity profile has a significant effect on the development of turbulent pulsations and, hence, of turbulent stresses in the flow.

As was shown in [1] and by the experimental data of [4, 5], in the case where the flows of forced and free convection in the layer near the wall are countercurrent ($R^* < 0$) there is additional disturbance of the layer at the wall and an increase in the turbulent viscosity in this layer. This leads to an increase in the resistance to the flow and an increase in heat transfer. Conversely, when the flows near the wall are coincident (case $R^* > 0$) the turbulent pulsations and, hence, the turbulent viscosity in this layer are greatly reduced. A change in the turbulent viscosity in the core of the flow has a very slight effect on the drag and heat transfer.

This effect can be called disturbance or stabilization of the viscous wall layer. We will try to take this effect into consideration by proceeding from the following considerations. Near the wall the additional turbulent viscosity ε/ν will depend on the dimensionless distance y_* from the wall in the fourth degree [6], and outside the wall layer ε/ν will become zero. This requirement is satisfied by the function

$$\varepsilon / \nu = a\,({}^1/_{11}\, y_*)^4 \exp\,(\,-{}^1/_{11}\, by_*)\,. \tag{1.5}$$

Constants a and b are determined from experiments.

An investigation of the experimental data of [5] shows that in the case where the flows of forced and free convection and the wall are coincident the heat transfer gradually decreases with increase in the free convection, the greatest reduction being 25–30% and occurring when the forced and free convection are of the same order. This can be used to determine a and b in (1.5). If we put $a = -4.4$, $b = 1.4$ and calculate the heat transfer, then the calculated Nusselt number will be 25% less than when $a = 0$.

When the velocity of free convection on the tube axis is the same as that of forced convection, we can, in view of the above-mentioned facts, put $a = -4.4$ and $b = 1.4$ with satisfactory accuracy if the flows at the wall are concurrent; $a = 4.4$, $b = 1.4$ if the flows at the wall are countercurrent. A graph of the function (1.5) for $a = 4.4$, $b = 1.4$ is shown by curve 4 in Fig. 1.

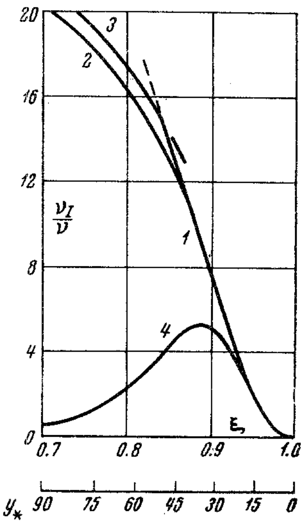

Fig. 1

Variables ξ and y_* are connected by the relationship $y_* = R_*(1-\xi)$. Figure 1, in addition to the scale for ξ, gives the additional scale for y_* for $R_* = 314.25$.

As is shown below (see ¶ 3), when $P = 1$, $P_t = 1$, $R_* = 314.25$, $Q_* = 0$, and free convection on the tube axis is equal to the forced convection, i.e.,

$$|\,u^{(1)}(0)\,| = |\,u^{(0)}(0)\,|$$

for $R_1^* = 1.1\cdot 10^5$ and $R_2^* = -4.2\cdot 10^4$. If $Q_* = 0.1R_*$, then this equality is fulfilled when $R_3^* = 1.7\cdot 10^5$; if $Q_* = -0.1R_*$, when $R_4^* = -1.90\cdot 10^4$. Figure 2 shows a plot of the coefficient a against R^* for these four cases. In each case we required that $a = 4.4$ for the indicated values of R^* and decreases by a factor of 10 if R^* increases or decreases by a factor of 10. The ascending branch of the curve was given by the function

$$a = 4.4\,\exp\,[-c_i\,(R^* - R_i^*)^4] \tag{1.6}$$

and the descending branch by the function

$$a = 4.4\exp\,[-c_i'\,(R^* - R_i^*)^2]. \tag{1.7}$$

The coefficients c_i and c_i' were determined so that (1.6) and (1.7) satisfied the above-indicated requirements. Expressions (1.6), (1.7) and coefficients a and b in (1.5) must be regarded as provisional and physical experiments will be required for their substantiation and accurate verification.

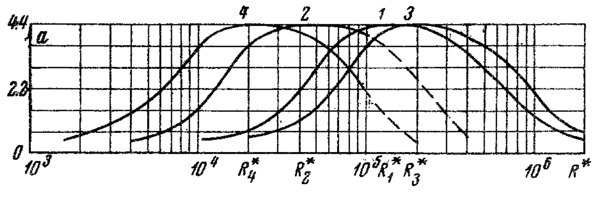

Fig. 2

In Fig. 2 curves are plotted for the parameters 1 ($R^* > 0$, $Q_* = 0$), 2 ($R^* < 0$, $Q_* = 0$), 3 ($R^* > 0$, $Q_* = 0.1R_*$), 4 ($R^* < 0$, $Q_* = -0.1R_*$). In every case $P = 1$, $P_t = 1$, $R_* = 314.25$.

After calculating $u^{(1)}(0)$ and $u^{(0)}(0)$ with allowance for the correction to the turbulent viscosity we must find the values of R_i^* and then determine the position of the curves in Fig. 2. Knowing R_i^*, we determine the correction to the turbulent viscosity by means of (1.5) and expressions (1.6) and (1.7). When $R^* > 0$, $Q_* = 0$, the coefficient $a < 0$; when $R^* < 0$, $Q_* = 0$, the coefficient $a > 0$. When $Q_* \neq 0$ we must use a more general rule: if the flows of forced and free convection at the wall are coincident, then $a < 0$; if the flows at the wall are countercurrent, then $a > 0$.

Equations (1.1) are weakly linear. The nonlinearity is due to the turbulent transfer coefficients. The principle of superposition of the solutions of the homogeneous and inhomogeneous equations, or of free and forced convection, can be applied to them only with some reservation. This principle is used below for the solution of the posed problem. Thus, some small error is introduced into the solution.

2. Theorems of momentum and kinetic energy. We derive integral relationships expressing the theorems of momentum and kinetic energy for combined turbulent forced and free convection in a vertical tube.

Integrating the first of equations (1.1) from zero to ξ and having in mind that the temperature in (1.1) must be measured relative to the mean temperature, we find the distribution of tangential stresses at a distance ξ from the axis:

$$\frac{\tau}{\rho_0} = -\left(g + \frac{1}{\rho_0}\frac{\partial p_0}{\partial z}\right)\frac{r_0}{2}\,\xi + \frac{G\nu^2}{r_0{}^2\xi}\int\limits_0^{\xi}(\theta^{(0,1)} - \langle\theta\rangle)\,\xi\, d\xi\,. \tag{2.1}$$

Using the second of equations (1.1) we find the temperature of the liquid, measured relative to the wall temperature at a distance ξ from the axis:

$$\theta^{(0,1)} - \theta_w = PR_* J_4\,(\xi,\,1). \tag{2.2}$$

In view of (2.2) we can bring (2.1) to the form

$$\frac{\tau}{\rho_0} = -\left(g + \frac{1}{\rho_0}\frac{\partial p_0}{\partial z}\right)\frac{r_0}{2}\,\xi - \frac{G\nu^2}{2r_0{}^3}\,(\langle\theta\rangle - \theta_w)\,\xi + \frac{GP\nu v_*\xi}{2r_0}J_4\,(\xi,\,1) - \frac{GP\nu v_*}{2r_0\xi}\,J_5\,(\xi,\,0). \tag{2.3}$$

Substituting (2.3) into the relationship

$$\frac{\tau}{\rho_0} = -\frac{\nu}{r_0}\left(1+\frac{\nu_t}{\nu}\right)\frac{dv}{d\xi} \qquad (2.4)$$

and carrying out several transformations, including integration by parts, we obtain

$$u^{(0)}(0) + u^{(1)}(0) = -\left[\left(g+\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}\right)\frac{r_0^2}{2\nu v_*} + \frac{GP\chi(\langle\theta\rangle - \theta_W)}{2r_0 v_*}\right]\int\limits_0^1 \frac{\xi\, d\xi}{1+\nu_t/\nu} - $$

$$-\frac{GP}{2}\int\limits_0^1\left[\frac{1}{\xi^2}\int\limits_0 \frac{\xi\, d\xi}{1+\nu_t/\nu} - \int\limits_1 \frac{d\xi}{(1+\nu_t/\nu)\,\xi}\right]\frac{J_3(\xi)\,\xi\, d\xi}{(1+\chi_t/\chi)}. \qquad (2.5)$$

Putting G = 0 in (2.5), we find the velocity on the tube axis in the case of forced convection

$$u^{(0)}(0) = R_*\int\limits_0^1 \frac{\xi\, d\xi}{1+\nu_t/\nu}. \qquad (2.6)$$

Here we take into consideration the equality

$$-\left(g+\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}\right) = \frac{2v_*^2}{r_0}. \qquad (2.7)$$

An analysis showed that with the adopted assumptions free convection will not create a flow of liquid through the cross section of the tube if the temperature is measured relative to the mean temperature given by the equality

$$\langle\theta^{(0,1)}\rangle - \theta_w = \left[\int\limits_0^1 (\theta^{(0;1)} - \theta_w)\, v^{(0)}\xi\, d\xi\right]\left(\int\limits_0^1 v^{(0)}\xi\, d\xi\right)^{-1}. \qquad (2.8)$$

Subtracting (2.6) from (2.5) and substituting the value $(\langle\theta\rangle - \theta_W)$, according to (2.8), after subjecting (2.8) to several transformations by using the second of Eqs. (1.1), we have

$$u^{(1)}(0) = GP\sqrt{\frac{f}{8}}\int\limits_0^1 \frac{J_1(\xi)J_3(\xi)\, d\xi}{(1+\chi_t/\chi)\,\xi} \times \int\limits_0^1 \frac{\xi\, d\xi}{1+\nu_t/\nu} - \frac{GP}{2}\int\limits_0^1\left[\frac{1}{\xi^2}\int\limits_0^\xi \frac{\xi\, d\xi}{1+\nu_t/\nu} - \int\limits_1^\xi \frac{d\xi}{(1+\nu_t/\nu)\,\xi}\right]\frac{J_3(\xi)\,\xi\, d\xi}{(1+\chi_t/\chi)}. \qquad (2.9)$$

The integral relationship (2.5) expresses the theorem of momentum. Equation (2.9), like (2.5), is valid for laminar, turbulent, and transitional flow regimes when the liquid does or does not contain internal heat sources, i.e., for all flow regimes for which Eqs. (1.1) are true.

We multiply the first of equations (1.1) by $v^{(0,1)}\xi d\xi$ and integrate it with respect to ξ from zero to unity and then, in view of the relationships

$$\int\limits_0^1 v^{(1)}\xi\, d\xi = 0, \quad \langle v\rangle = 2\int\limits_0^1 v^{(0)}\xi\, d\xi,$$

$$\int\limits_0^1 v^{(0)}(\theta^{(0,1)} - \langle\theta\rangle)\,\xi\, d\xi = 0 \qquad (2.10)$$

and (2.7) we have

$$\frac{1}{2}R_* R + G\int\limits_0^1 (\theta^{(0,1)} - \langle\theta\rangle)\, u^{(1)}\xi\, d\xi - R_*\int\limits_0^1\left(1+\frac{\nu_t}{\nu}\right)\left(\frac{du^{(0,1)}}{d\xi}\right)^2\xi\, d\xi = 0. \qquad (2.11)$$

The last of relationships (2.10) can easily be obtained from (2.8).

Substitution of (2.2) into (2.11) and the performance of several transformations gives

$$\frac{R}{2} - GP\int\limits_0^1 \frac{J_8(\xi)J_3(\xi)\, d\xi}{(1+\chi_t/\chi)\,\xi} - \int\limits_0^1\left(1+\frac{\nu_t}{\nu}\right)\left(\frac{du^{(0,1)}}{d\xi}\right)^2\xi\, d\xi = 0. \qquad (2.12)$$

The integral relationship (2.12), like (2.5), is valid for all flow regimes for which Eqs. (1.1) are true. The terms of Eq. (2.12) with accuracy to the constant factor express the work of external pressure forces, the Archimedean upthrust, and viscosity forces, respectively.

In the case of pure forced convection (2.12) takes the form

$$\frac{R}{2} - \int\limits_0^1\left(1+\frac{\nu_t}{\nu}\right)\left(\frac{du^{(0)}}{d\xi}\right)^2\xi\, d\xi = 0. \qquad (2.13)$$

Subtracting (2.13) from (2.12) and taking into account that

$$\int\limits_0^1\left(1+\frac{\nu_t}{\nu}\right)\frac{du^{(0)}}{d\xi}\frac{du^{(1)}}{d\xi}\xi\, d\xi = 0$$

we find

$$-GP\int\limits_0^1 \frac{J_8(\xi)J_3(\xi)\, d\xi}{(1+\chi_t/\chi)\,\xi} - \int\limits_0^1\left(1+\frac{\nu_t}{\nu}\right)\left(\frac{du^{(1)}}{d\xi}\right)^2\xi\, d\xi = 0, \qquad (2.14)$$

3. Velocity distribution. We use the above-derived integral relationships to determine the velocity in the case of combined forced and free convection. We first find the velocity for forced convection.

If free motion is imposed on the forced motion, the turbulent viscosity is altered and, hence, the velocity is also altered. For instance, $u^0(0) = 20$ when $R_* = 314.25$ ($R = 10^4$). If we put $a = -4.4$ and $b = 1.4$ in (1.5) and carry out the corresponding calculations we obtain $u^{(0)}(0) = 35.0$, which is 75% greater than the previous value. If we take $a = 4.4$, $b = 1.4$, we obtain $u^{(0)}(0) = 16.0$, i.e., 20% smaller.

The logarithmic formulas and the one-seventh law, which satisfactorily represent the velocity distribution in the core of a turbulent flow in a tube, become invalid close to the wall. The formula determined in [3], which is true over the whole cross section of the tube, is cumbersome and, hence, is not at all convenient to work with. Below we obtain a simple and convenient formula which provides a good representation of the velocity distribution over the whole cross section of the tube, including the immediate vicinity of the wall, and which satisfies the boundary condition on the wall.

We will seek the velocity profile in the tube for turbulent flow in the form

$$u^{(0)} = a_1\left[(1-\xi^n) + \frac{1}{3}(1-\xi^2)\right], \qquad (3.1)$$

where n and a_1 are undetermined coefficients. To determine n we calculate the mean velocity $\langle v\rangle$ over the cross section and the tangential stress τ_w on the wall by using (3.1) and substitute the values in the expression for the drag coefficient $f \equiv 8\tau_w/\rho\langle v\rangle^2$. On the other hand, the value of the drag coefficient can be found from the empirical Blasius formula

$$f = 0.316\ R^{-1/4} \qquad (2.3\ 10^3 \leqslant R \leqslant 10^5) \qquad (3.2)$$

or the Filipenko formula [7], which is suitable for a wide range of Reynolds numbers,

$$f = (1.82\ \lg R - 1.64)^{-2}. \qquad (3.3)$$

Equating these values for the drag coefficient we find

$$n = {}^1/_6\,[(7k-8) \pm \sqrt{(7k-4)^2 - 32k}]\,. \qquad (3.4)$$

Here

$$k = 0.00988\,R^{3/4} \quad \text{or} \quad k = {}^1/_{32}\,R\,(1.82\ \lg R - 1.64)^{-2}, \qquad (3.5)$$

respectively, for f according to (3.2) or (3.3).

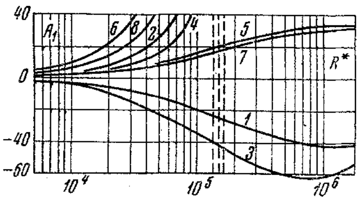

Fig. 3

If the turbulent viscosity in the tube is known there is no need to use empirical data for f. The drag coefficient can be determined from the formula derived below (see ¶ 4).

To determine a_1 we can use relationship (2.6); we determine $u^{(0)}_{max}$ from (2.6) and find a_1 from the equality

$$^4/_3\,a_1 = u^{(0)}_{max}\,. \qquad (3.6)$$

As heat transfer calculations show, (3.1) is almost as accurate as the formula from [3]. It follows from (3.4) that if R = 10^4, then n = 20.6; if R = $2.5 \cdot 10^4$, then n = 43; if R = 10^5, then n = 128.

We find the velocity distribution for combined laminar forced and free convection. The functions in the form of which we seek the solution must satisfy the boundary condition of attachment of the liquid to the wall and the condition of closure of the free-convection flow. As an approximating function we take

$$u^{(1)} = A_1(1-\xi^2)\ (1-3\xi^2) + A_2(1-\xi^{10})\ (1-{}^7/_3\ \xi^2). \qquad (3.7)$$

Here A_1 and A_2 are unknown coefficients. We substitute function (3.7) and

$$u^{(0)} = {}^1/_2\,R_*\,(1-\xi^2) \qquad (3.8)$$

into (2.9) and (2.14) after putting $\nu_t/\nu = 0$ in them, and from these two equations we find A_1 and A_2. Such calculations show that the coefficient A_1 must be found from a quadratic equation containing the parameters GP, Q_*, and R_*. Then A_2 is determined from a linear equation. Since the equation for A_1 is quadratic, then for given parameters GP and Q_* we obtain two values of A_1 and then we find two values of A_2. One pair of the coefficients A_1 and A_2 (we will call it the main pair) gives practically complete agreement with the exact solution [8, 9]. The second (subsidiary) pair gives poor agreement with the solution in [8, 9]. Of the two coefficients A_1 we must take the smaller in modulus as the main one. A comparison with the exact solution where the values of the parameters are $-100 \leq R^* \leq 10^4$,

$$Q_* = 0, \quad Q_* = {}^1/_4\,R_*, \quad Q_* = {}^1/_{40}\,R_*, \quad Q_* = {}^5/_2\,R_*$$

shows that the greatest deviation from the accurate result does not exceed 0.5%. In most cases the deviation is much smaller.

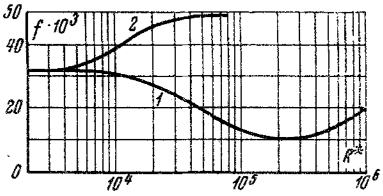

Fig. 4

Since this method of solution has proved satisfactory we find the velocity for combined turbulent forced and free convection. The approximating function for turbulent free convection must satisfy the boundary condition and the condition of closure of the free-convection flow; it must also reflect the presence of a large velocity gradient at the wall. We put

$$u^{(1)} = A_1\,(1-\xi^{30})\,(1-{}^{17}/_8\,\xi^2). \qquad (3.9)$$

As $u^{(0)}$ we use formula (3.1). The drag coefficient f is determined from the formula derived below (see ¶ 4). The integrals contained in (2.9) and (2.14) are calculated by numerical integration using (1.3) and (1.4). The results of the calculations are as follows.

With $R_* = 314.25$, $P_t = 1$, $P = 1$ we calculated $u^{(1)}(0) = A_1$ by means of (2.9) and (2.14). The formulas are of the same type and the coefficients in them do not differ by more than 5%. The arithmetic mean of these coefficients is taken. In this way we find

$$A_1 = \frac{(-3.20\ 10^{-4} + 1.80\ 10^{-5}\,Q_*)\,GP}{0.7\ 10^{-5}\,GP + 1}. \qquad (3.10)$$

So far we have ignored disturbance or stabilization of the layer near the wall. We will now take this into account. The coefficient A_1 is calculated by using (2.9), (2.14), (1.3), (1.4) and (1.5) with $R_* = 314.25$, $P_t = 1$, $P = 1$, $a = 4.4$, $b = 1.4$,

$$A_1 = \frac{(-2.10\cdot10^{-4} + 1.35\cdot10^{-5}\,Q_*)\,GP}{0.6\cdot10^{-5}GP + 1}. \qquad (3.11)$$

With $R_* = 314.25$, $P_t = 1$, $P = 1$, $a = -4.4$, $b = 1.4$ we have

$$A_1 = \frac{(-6.20\cdot10^{-4} + 2.60\cdot10^{-5}Q_*)\,GP}{0.8\cdot10^{-5}GP + 1}. \qquad (3.12)$$

Formulas (3.11) and (3.12) are true only for particular values of R^*, since for each R^* there is a particular a (Fig. 2). For other a we carried out similar calculations and obtained subsidiary formulas,

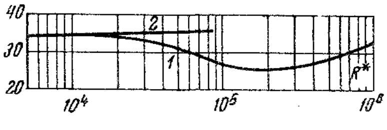

Fig. 5

which are also true for particular R^*. In this way we find several values of A_1, depending on R^*. Figure 3 shows the relationship $A_1 = A_1(R^*)$ for different combinations of values of the parameters:

1 $(R^* > 0,\ Q_* = 0,\ a = 0)$,

2 $(R^* < 0,\ Q_* = 0,\ a = 0)$,

3 $(R^* > 0,\ Q_* = 0,\ a < 0)$.

4 $(R^*<0,\ Q_*=0,\ a>0)$,

5 $(R^*>0,\ Q_*=0.1\,R_*,\ a=0)$,

6 $(R^*<0,\ Q_*=-0.1\,R_*,\ a=0)$,

7 $(R^*>0,\ Q_*=0.1\,R_*,\ a>0)$,

8 $(R^*<0,\ Q_*=-0.1\,R_*,\ a>0)$.

We took into account the relationship between a and R^* according to Fig. 2. In every case $P = 1$, $P_t = 1$, $R_* = 314.25$. In the deduction of (3.10)–(3.12) we took $Q_* =$ const.

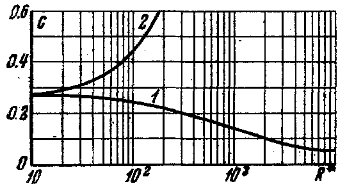

Fig. 6

4. Drag law. We determine the drag law for combined forced and free convection in a vertical tube. The mean velocity over the cross section of the tube will be

$$\langle v\rangle = 2\int_0^1 v^{(0,1)}\xi\,d\xi\,. \tag{4.1}$$

Performing several transformations and using (2.4), (2.7), and the first of equalities (2.10), we find

$$\langle v\rangle = -\frac{r_0^2}{2\nu\rho_0}\left(\rho_0 g + \frac{\partial p_0}{\partial z}\right)\int_0^1 \frac{\xi^3\,d\xi}{1+\nu_t/\nu}. \tag{4.2}$$

According to the definition of the drag coefficient, we have

$$f \equiv -\frac{2r_0}{{}^1/_2\rho_0\langle v\rangle^2}\left(\rho_0 g + \frac{\partial p_0}{\partial z}\right). \tag{4.3}$$

Eliminating the factor in the parentheses from (4.2) and (4.3) we obtain

$$\frac{1}{f} = \frac{R}{16}\int_0^1 \frac{\xi^3\,d\xi}{1+\nu_t/\nu}\,. \tag{4.4}$$

It follows from (4.3) and (2.7) that

$$R_* = R\sqrt{{}^1/_{32}f}\,. \tag{4.5}$$

If we eliminate the Reynolds number R from (4.4) and (4.5) we find

$$f = 8\,/\,J^2R_*^{\ 2}\,, \tag{4.6}$$

where J denotes the integral contained in (4.4). If the parameter R_* or $(\rho_0 g + \partial p_0/\partial z)$ is prescribed, we find the integral J by numerical integration and then, according to (4.6), we find f. After finding J and f, we can determine R from (4.4) or (4.5).

Equality (4.4) is valid for laminar, turbulent, and transitional flow regimes. In the case of laminar flow we can derive the Poiseuille formula from (4.4):

$$1\,/\,f = {}^1/_{64}\,R. \tag{4.7}$$

We calculated f and R for $R_* = 314.25$, using (1.3) and (1.4), and found that $f = 0.0316$, $R = 10^4$. If we take into account disturbance or stabilization of the wall layer according to (1.5), then for $R_* = 314.25$, $a = 4.4$, $b = 1.4$ we have $f = 0.0495$, $R = 8000$, i.e., in this case R is 20% less than when $a = 0$. With the same R_* and $a = -4.4$, $b = 1.4$ the calculations give: $f = 0.0102$, $R = 17\,600$, i.e., R is increased by 76%.

Thus, free convection affects the drag through disturbance or stabilization of the wall layer and this effect is fairly significant. In the case of turbulent flow in a vertical tube the drag can be regulated by heating or cooling the wall.

Figure 4 shows f as a function of R^* for $R_* = 314.25$, $Q_* = 0$ with due regard to (1.3), (1.4), (1.5), and Fig. 2. Curve 1 is for $R^* > 0$ and curve 2 for $R^* < 0$.

5. Heat transfer in absence of internal heat sources. We calculate the heat transfer in the case of combined turbulent forced and free convection in a vertical tube without internal heat sources in the liquid.

By definition the Nusselt number is

$$N \equiv \frac{2r_0 q_w}{\lambda(\langle T\rangle - T_w)}\,. \tag{5.1}$$

Using the second of Eqs. (1.1) we find the heat flux density q_w on the wall for $Q_* = 0$:

$$q_w = -\,{}^1/_2\,\rho C_p A r_0\,\langle v\rangle\,. \tag{5.2}$$

Substituting into (5.1) equality (5.2) and the expression for the mean temperature according to heat content

$$\langle T\rangle - T_w = 2\int_0^1 (T - T_w)\,w^{(0,1)}\xi\,d\xi \tag{5.3}$$

and taking into account the relationship

$$v_* = \langle v\rangle\sqrt{f/8}\,, \tag{5.4}$$

we obtain

$$\frac{1}{N} = 2\int_0^1 \frac{J_6^2(\xi)\,d\xi}{(1+P_1\nu_t/\nu)\,\xi}\,,\quad\text{or}\quad \frac{1}{N} = \frac{f}{4}\int_0^1 \frac{J_2^2(\xi)\,d\xi}{(1+P_1\nu_t/\nu)\,\xi}. \tag{5.5}$$

Equality (5.5) is valid for all flow regimes for which Eqs. (1.1) are true in the case $Q_* = 0$.

Using (5.5) we can find how the heat transfer is affected by the free motion superimposed on the forced motion. We consider first the combination of laminar forced and free convection.

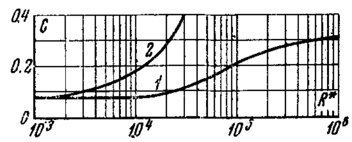

Fig. 7

If the flows of forced and free convection at the wall are countercurrent ($R^* < 0$), then the mean temperature of the liquid increases [in formula (5.5) in this case $J_8(\xi) > 0$], which, as (5.1) shows, leads to a reduction of N. If the flows at the wall are coincident ($R^* > 0$), then the mean temperature of the liquid decreases [in (5.5) $J_8(\xi) < 0$], which leads to an increase of N. At the same time, free convection superimposed on forced convection, with the adopted assumptions (averaged velocity parallel to tube axis, constant vertical temperature gradient on walls), does not affect the heat flux density on the wall. This is apparent from (5.2) at least [see also [8], Chap. 3, §5].

The relationship between the Nusselt number N and the Rayleigh number R^* was calculated from (5.5) for $R^* > 0$ and $R^* < 0$. We used the formula $f = 64R^{-1}$ or the relationship derived from it and

(5.4), viz., $2R = R_*^2$, which is valid for laminar forced flow, and also for combined laminar forced and free flow (¶ 4). The relationship between N and R* calculated in this way agrees almost completely with that found in [9, 10]. The curves representing this relationship for $R^* > 0$ and $R^* < 0$ agree satisfactorily with the experimental data given in [9, 10]. In the case of turbulent motion as (5.5) shows, free convection affects the heat transfer in two ways.

If the flows of forced and free convection at the wall are countercurrent ($R^* < 0$), the turbulent viscosity at the wall is increased and this intensifies heat transfer. On the other hand, in this case the mean temperature of the liquid is increased, which leads to a reduction of N. Calculation or experiment would show which of these factors predominates.

If the flows at the wall are coincident ($R^* > 0$) the turbulent viscosity at the wall is reduced and N is also reduced. In this case, however, the mean temperature of the liquid is reduced, which leads to an increase in N.

We calculated the Nusselt number for $R_* = 314.25$, $P_t = 1$, $P = 1$, $b = 1.4$, $a = 4.4$ (the flows at the wall are countercurrent). The result was as follows: $N = 34.9$, i.e., 1% greater than when $a = 0$. When $R_* = 314.25$, $P_t = 1$, $P = 1$, $b = 1.4$, $a = -4.4$ (the flows at the wall are coincident) the number $N = 25.9$, i.e., 25% less than when $a = 0$.

Figure 5 shows N as a function of R*; curve 1 is for the case $R^* > 0$ and curve 2 for $R^* < 0$. In the calculations we took into account (1.3), (1.4), (1.5), (1.6), and (1.7).

A comparison of the heat transfer in laminar and turbulent flow regimes clearly shows that in the case $R^* > 0$ free convection intensifies heat transfer in a laminar flow and reduces it in a turbulent flow. In the case $R^* < 0$ free convection reduces heat transfer in a laminar flow and increases it in a turbulent flow. This can be attributed to the fact that in the case of laminar flow one factor due to free convection affects the heat transfer, whereas in the case of turbulent flow there is a second effect (disturbance or stabilization of the wall layer), and this second effect predominates.

When the parameter R_* has values greater than 314.25, free convection will be comparable with forced convection at higher Rayleigh numbers than when $R_* = 314.25$. Hence, for R_* greater than 314.25, the curves $N = N(R^*)$ will be like that illustrated in Fig. 5, but will be shifted upwards and to the right.

6. Heat transfer in presence of internal heat sources. In [11] an integral Lyon relationship for calculating heat transfer in the absence of internal heat sources in the liquid was extended to the case of a liquid containing heat sources. The flow of liquid in the tube was assumed to be purely forced. We extend these integral relationships to combined forced and free convection in a vertical tube.

Integrating the energy equation—the second of Eqs. (1.1)—in the case of uniformly distributed internal heat sources of power Q and constant heat flux density q_w on the wall we find

$$T - T_w = -\frac{2q_w r_0}{\lambda}\int_1^{\xi}\frac{J_6(\xi)\,d\xi}{(1+\chi_t/\chi)\,\xi} + \frac{Qr_0^2}{\lambda}\int_1^{\xi}\frac{J_7(\xi)\,d\xi}{(1+\chi_t/\chi)\,\xi}. \tag{6.1}$$

Substituting (6.1) into the expression for the mean temperature according to heat content (5.3) and performing several transformations we obtain

$$\langle T\rangle - T_w = \frac{4q_w r_0}{\lambda}\int_0^1\frac{J_6^2(\xi)\,d\xi}{(1+\chi_t/\chi)\,\xi} - \frac{2Qr_0^2}{\lambda}\int_0^1\frac{J_6(\xi)\,J_7(\xi)\,d\xi}{(1+\chi_t/\chi)\,\xi}. \tag{6.2}$$

We put $q_w = 0$ in (6.2) and then we have

$$\langle T\rangle - T_{aw} = -\frac{2Qr_0^2}{\lambda}\int_0^1\frac{J_6(\xi)\,J_7(\xi)\,d\xi}{(1+\chi_t/\chi)\,\xi}, \tag{6.3}$$

where T_{aw} denotes the adiabatic wall temperature.

If we assume the density of the internal heat sources to be arbitrary over the cross section of the tube, similar calculations will give

$$\langle T\rangle - T_{aw} = -\frac{2r_0^2}{\lambda}\int_0^1\frac{1}{(1+\chi_t/\chi)\,\xi}J_6(\xi)\times \left[2J_6(\xi)\int_0^1 Q\xi\,d\xi - \int_0^{\xi} Q\xi\,d\xi\right]d\xi. \tag{6.4}$$

We now subtract (6.2) from (6.3)

$$T_w - T_{aw} = -\frac{4q_w r_0}{\lambda}\int_0^1\frac{J_6^2(\xi)\,d\xi}{(1+\chi_t/\chi)\,\xi}. \tag{6.5}$$

It is clear from (6.5) that when the liquid contains internal heat sources the heat flux density q_w on the wall will be proportional to the temperature difference $(T_{aw} - T_w)$, and not the difference $(\langle T\rangle - T_w)$, as is the case in the absence of sources. From (6.5) we have

$$\frac{1}{N^*} = 2\int_0^1\frac{J_6^2(\xi)\,d\xi}{(1+\chi_t/\chi)\,\xi}, \quad N^* \equiv \frac{q_w 2r_0}{(T_{aw} - T_w)\,\lambda}. \tag{6.6}$$

Here N* is the Nusselt number.

If the heat transfer coefficient α is referred, as usual, to the temperature difference $(\langle T\rangle - T_w)$, the Nusselt number will be

$$\frac{1}{N} = 2\int_0^1\frac{J_6^2(\xi)\,d\xi}{(1+\chi_t/\chi)\,\xi} - 2z_1\int_0^1\frac{J_6(\xi)\,J_7(\xi)\,d\xi}{(1+\chi_t/\chi)\,\xi}\quad\left(z_1 = \frac{Qr_0}{2q_w}\right). \tag{6.7}$$

Here z_1 is the relative density of the internal heat sources.

At a certain value of z_1 the Nusselt number, determined from (6.7), will become infinite. Hence, in the case of internal heat sources in the liquid the heat transfer will be characterized not by N, but by the ratio [11]

$$-\frac{(T_{aw} - \langle T\rangle)}{(T_{aw} - T_w)} = z_1 N_0 \Delta T_a,$$

$$\Delta T_a = \frac{\lambda}{Qr_0^2}(\langle T\rangle - T_{aw}). \tag{6.8}$$

Here ΔT_a is the dimensionless difference between the mean calorimetric temperature of the liquid and the adiabatic wall temperature; N_0 is the Nusselt number for a flow without internal sources. In [11] the relationship between this quantity and the Reynolds and Prandtl numbers was calculated for pure forced turbulent flow. Below we give the results of calculations of the relationship between this quantity and the Rayleigh number in the case of combined forced and free convection in a vertical tube.

Figure 6 shows

$$C \equiv -\frac{(T_{aw} - \langle T \rangle)}{(T_{aw} - T_w) Z_1} \tag{6.9}$$

as a function of R^* for combined laminar forced and free convection in a vertical tube with $Q_* = \pm R_*/4$; curve 1 is for $R^* > 0$, $Q_* = R_*/4$ and curve 2 for $R^* < 0$, $Q_* = -R_*/4$. We note that $Q_* < 0$ when $R^* < 0$, although the heat sources are positive ($Q > 0$). When $R^* \to 0$, C tends to the limit 3/11, corresponding to pure forced laminar flow. Since Q_* is prescribed, then z_1 will be determined from the relationship

$$z_1/(z_1 - 1) = Q_* \sqrt{1/8 f}\,. \tag{6.10}$$

In the deduction of (6.10) we used (6.11). Relationship (6.10) applies to both laminar and turbulent flows.

In the case of combined forced and free convection, as distinct from pure forced convection, the source density Q_* has to be prescribed if the coefficients in the expression for the free convection are to be found.

The quantity C is equal to the ratio of the integrals contained in expression (6.7). In combined turbulent forced and free convection C is affected by the same two factors which affect N in the case of the absence of internal sources. If the heat flux density is positive and sufficiently high (for instance, $Q_* = 0.1R_*$), then the liquid in the center of the tube will be hotter than at the wall and, hence, when $\beta > 0$ free convection will be upward in the center of the tube and downward at the wall, irrespective of the sign of the Rayleigh number. If the forced convection of the liquid is upward ($\langle v \rangle > 0$), then at the wall the flows will be countercurrent. This is obvious from expression (3.10) and from Fig. 3.

Taking into account the correction to the turbulent viscosity, according to (1.5)–(1.7), we calculated f, R, a_1, n, and A_1 from the corresponding formulas, and then the value of C from (6.9) for $P = 1$, $P_t = 1$, $R_* = 314.25$, $Q_* = \pm 0.1R_*$. Figure 7 shows the calculated relationship between C and R^*. Curve 1 is for $R^* > 0$ and $Q_* = 0.1R_*$; curve 2 is for $R^* < 0$ and $Q_* = -0.1R_*$. In both cases the flows at the wall are countercurrent and, hence, in both case $a > 0$ in expression (1.5).

We find the density of internal heat sources in the flow of a liquid through a tube without heat transfer—the walls are thermally insulated. In this case the quantity of heat from the sources per unit volume of the liquid is equal to the amount of heat removed by the flow from unit volume.

Integration of the energy equation with Q = const gives

$$1/2\, \rho c_p A r_0 \langle v \rangle = -q_w + 1/2 Q r_0. \tag{6.11}$$

Putting $q_w = 0$ and using (5.4) we find

$$Q_* = \sqrt{8/f}\,. \tag{6.12}$$

The drag coefficient f is found from formula (4.6).

For laminar flow—forced convection or combined forced and free convection—(4.7) is valid. Then (6.12) gives

$$Q_* = \sqrt{1/8 R} = 1/4 R_*. \tag{6.13}$$

In the case of turbulent flow for $R_* = 314.25$, according to (4.6), we have $f = 0.0316$ and, using (6.12), we find $Q_* = 15.9$, i.e., approximately $R_*/19.75$.

Thus, the procedure for the solution of the posed problem is as follows. From the given value of dp_0/dz we find R_* from expression (2.7). Using the parameter R_* we find f and R from formulas (4.4)–(4.6) and $u^{(0)}(0)$ from formula (2.6). From the found value of the Reynolds number R we determine the index n from (3.4). Knowing $u^{(0)}(0)$, we calculate a_1 from (3.6). From the prescribed parameters P, P_t, and R_* we find the coefficient A_1, i.e., an expression of the form (3.10), by using the integral relationships (2.9) or (2.14). We find the correction to the turbulent viscosity due to disturbance or stabilization of the viscous wall layer by menas of (1.5) for any a, e.g., $a = 4.4$. Using this correction we again find f, R, a_1, and n from the above-mentioned formulas and calculate expressions of the form (3.11), each of which is true only for a particular value of R^*, since for each R^* there is a particular a. Using an expression of the form (3.11) with prescribed Q_* we determine the value of the Rayleigh number R_i^* for which free convection is equal to the forced convection, i.e., $u^{(1)}(0) = u^{(0)}(0)$; we determine the position of the curves in Fig. 2. In this way we find several values of A_1 in relation to R^*.

To determine the heat transfer in the case of absence of internal heat sources we calculate the integral (5.5) with introduction of the correction to the turbulent viscosity. In the case of the presence of internal heat sources we must first find the correction to the turbulent viscosity from the prescribed P, P_t, R_*, and Q_*, as in the case $Q_* = 0$; we then find C, which characterizes the heat transfer in the case $Q_* \neq 0$, from (6.9).

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REFERENCES

1. L. E. Ber, "Solution of the problem of combined turbulent forced and free heat convection in a vertical tube with internal heat sources present in the liquid," Izv. AN SSSR. OTN, Mekhanika i mashinostroenie, no. 6, 1962.

2. L. E. Ber, "Solution of equations of mixed turbulent convection in a vertical tube," Izv. AN SSSR. OTN, Mekhanika i mashinostroenie, no. 3, 1963.

3. H. Reichardt, "Vollständige Darstellung der turbulenten Geschwindigkeitsverteilung in glatten Leitungen," Z. angew. Math. und Mech., vol. 31, p. 208, 1951.

4. B. S. Petukhov and L. D. Nol'de, "Heat transfer in a viscous gravitational flow of liquid in tubes," Teploenergetika, no. 1, 1959.

5. E. R. G. Eckert and A. I. Diaguila, "Convective heat transfer for mixed, free, and forced flow through tubes," Trans. ASME, vol. 76, no. 4, 1954.

6. L. G. Loitsyanskii, "Heat transfer in turbulent flow," PMM, vol. 24, no. 4, 1960.

7. G. K. Filonenko, "Hydraulic resistance of pipes," Teploenergetika, no. 4, 1954.

8. G. A. Ostroumov, Free Convection in the Conditions of the Internal Problem [in Russian], Gostekhizdat, 1952.

9. T. M. Hallman, "Combined forced and free laminar heat transfer in vertical tubes with uniform internal heat generation," Trans. ASME, vol. 78, no. 8, 1956.

10. W. G. Brown, "Die Überlagerung von erzwungener und natürlicher Konvektion in senkrechten Röhren ..." VDI—Forschungsheft, no. 480, 1960.

11. B. S. Petukhov, L. G. Genin, and V. L. Mal'ter, "Heat transfer in tubes with internal heat sources present in the liquid," Inzh.-fiz. zh., vol. 3, no. 9, 1960.

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